

A short introduction to models and inference for spatial point processes.

Frédéric Lavancier



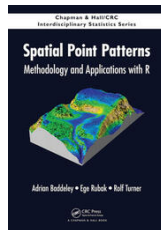
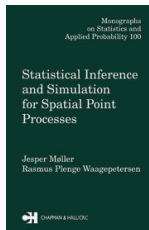
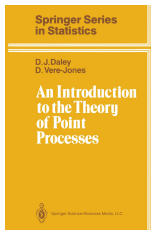
1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

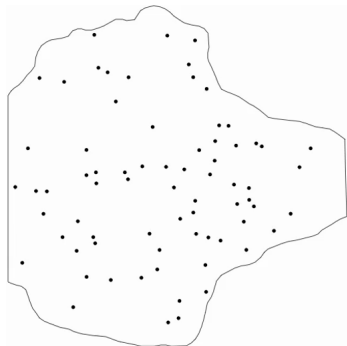


- D.J Daley and D. Vere-Jones. *An introduction to the Theory of Point Processes*. Springer Series in Statistics, 1998.
- J. Møller and R. P. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall/CRC, Boca Raton, 2004.
- A. Baddeley, E. Rubak, and R. Turner. *Spatial Point Patterns : Methodology and Applications with R*. CRC Press, 2015.

All illustrations have been implemented with the **spatstat library** in **R**. Almost all presented datasets also come from this library.

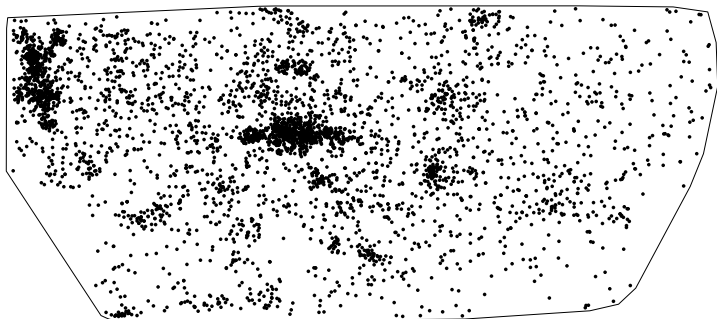
Some examples of point patterns: 2D

Locations of some (Langerin) **proteins** in a living cell.



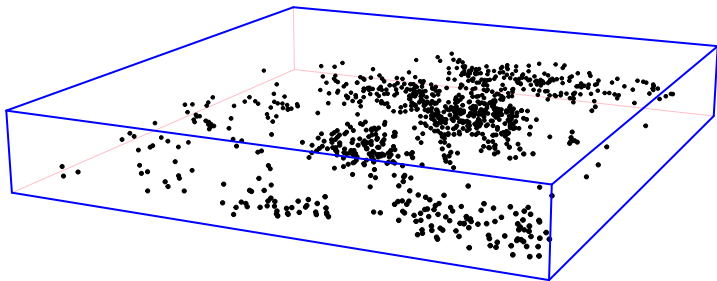
Some examples of point patterns: 2D

Positions of 4215 **galaxies** in the Shapley Supercluster.



Some examples of point patterns: 3D

Locations of **nucleosomes** in a cell nucleus of a brain.

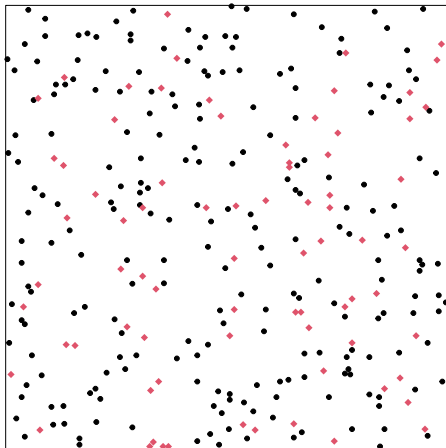


Some examples of point patterns: with discrete marks

Cell nuclei in hamster kidney, subject to a metastatic lymphoma.

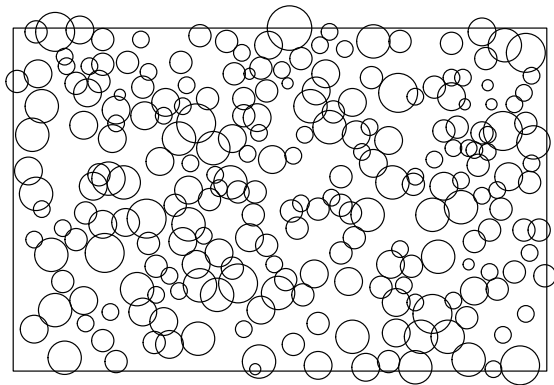
In black: “dividing” cells

In red: “pyknotic” cells, i.e. dying cells



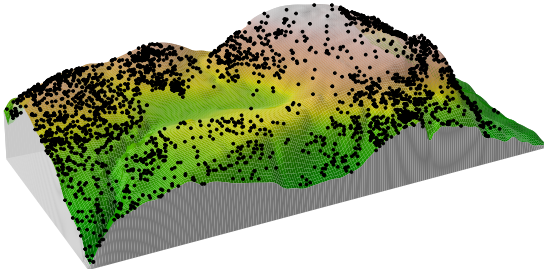
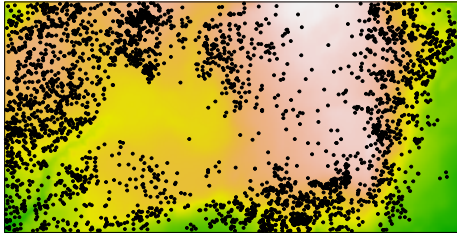
Some examples of point patterns: continuous marks

Locations and diameters of **sea anemones** (at Quiberon)



Some examples of point patterns: with auxiliary information

Locations of **trees** in a tropical rain forest.
Auxiliary information: **elevation** in the study region



In this presentation, for simplicity:

- we work on a **bounded** set $S \subset \mathbb{R}^d$ (typically $d = 2$)
- the point patterns are **simple** (no duplicated points)
- there is **no mark** (neither discrete, nor continuous)
- there are **no auxiliary information** (no covariate)
- we do not consider temporal, nor space-time, point processes

The main goal is to analyse/model the repartition of points.

What is a spatial point process?

A **point pattern** \mathbf{x} in S is a locally finite subset of S .

Denoting by $n(A)$ the cardinality of any set $A \subseteq S$, this means:

$$\begin{cases} \mathbf{x} \subset S, \\ \text{for all } A \subseteq S, \ n(\mathbf{x} \cap A) < \infty. \end{cases}$$

Hence, it makes sense to write $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$ where $n(\mathbf{x}) = n(\mathbf{x} \cap S)$ and $x_i \in S$ for all $i = 1, \dots, n(\mathbf{x})$.

What is a spatial point process?

A **point pattern** \mathbf{x} in S is a locally finite subset of S .

Denoting by $n(A)$ the cardinality of any set $A \subseteq S$, this means:

$$\begin{cases} \mathbf{x} \subset S, \\ \text{for all } A \subseteq S, \ n(\mathbf{x} \cap A) < \infty. \end{cases}$$

Hence, it makes sense to write $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$ where $n(\mathbf{x}) = n(\mathbf{x} \cap S)$ and $x_i \in S$ for all $i = 1, \dots, n(\mathbf{x})$.

A **spatial point process** \mathbf{X} is a **random point pattern** in S .

- The number of points $n(\mathbf{X})$ is random.
- The locations of the points in S are random.

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

The first order intensity ρ answers two questions:

- How many points can be expected?
- Where are the points more likely to occur?

First order intensity ρ

The first order intensity ρ answers two questions:

- How many points can be expected?
- Where are the points more likely to occur?

By definition, the intensity ρ satisfies:

$$\forall A \subseteq S, \quad \mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E} \left(\sum_{u \in \mathbf{X}} \mathbf{1}_{u \in A} \right) = \int_A \rho(u) du.$$

Intuitively, for $u \in S$, $\rho(u) \approx \mathbb{P}(\mathbf{X} \text{ has a point at } u)$.

In particular: $\mathbb{E}(n(\mathbf{X})) = \int_S \rho(u) du$.

An important special case : the **homogeneous** case

$$\rho(u) = \rho \quad \text{for any } u \in S.$$

Then,

- for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \rho|A|$
- the points are equally likely to appear anywhere.

First order intensity ρ

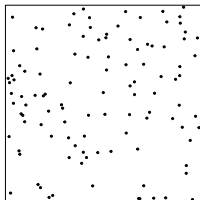
An important special case : the **homogeneous** case

$$\rho(u) = \rho \quad \text{for any } u \in S.$$

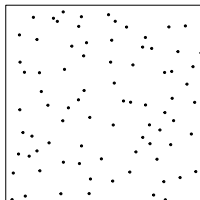
Then,

- for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \rho|A|$
- the points are equally likely to appear anywhere.

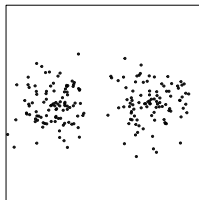
Otherwise, the point process is **inhomogeneous**.



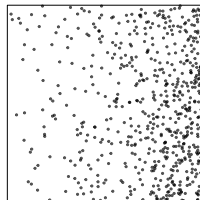
Homogeneous



Homogeneous



Inhomogeneous



Inhomogeneous

First order intensity ρ : estimation

Assume we observe a realisation \mathbf{x} of \mathbf{X} in S . How can we estimate ρ ?

First order intensity ρ : estimation

Assume we observe a realisation \mathbf{x} of \mathbf{X} in S . How can we estimate ρ ?

If \mathbf{X} is assumed to be homogeneous : $\hat{\rho} = n(\mathbf{x})/|S|$.

First order intensity ρ : estimation

Assume we observe a realisation \mathbf{x} of \mathbf{X} in S . How can we estimate ρ ?

If \mathbf{X} is assumed to be homogeneous : $\hat{\rho} = n(\mathbf{x})/|S|$.

If \mathbf{X} is **not** assumed to be homogeneous:

- non parametric estimation of $u \mapsto \rho(u)$: for a density kernel k (typically the standard Gaussian density), and a bandwidth $h > 0$,

$$\hat{\rho}(u) = \sum_{v \in \mathbf{x}} \frac{1}{h^d} k\left(\frac{v-u}{h}\right) / K_h(v), \quad u \in S,$$

where $K_h(v) = h^{-d} \int_S k\left(\frac{v-u}{h}\right) du$ accounts for edge effects ($K_h(v) \approx 1$ if v is far from the border of S).

Remark: $\int_S \hat{\rho}(u) du = n(\mathbf{x})$ in agreement with $\int_S \rho(u) du = \mathbb{E}(n(\mathbf{x}))$.

The choice of h is crucial: adaptive choices are available.

First order intensity ρ : estimation

Assume we observe a realisation \mathbf{x} of \mathbf{X} in S . How can we estimate ρ ?

If \mathbf{X} is assumed to be homogeneous : $\hat{\rho} = n(\mathbf{x})/|S|$.

If \mathbf{X} is **not** assumed to be homogeneous:

- non parametric estimation of $u \mapsto \rho(u)$: for a density kernel k (typically the standard Gaussian density), and a bandwidth $h > 0$,

$$\hat{\rho}(u) = \sum_{v \in \mathbf{x}} \frac{1}{h^d} k\left(\frac{v-u}{h}\right) / K_h(v), \quad u \in S,$$

where $K_h(v) = h^{-d} \int_S k\left(\frac{v-u}{h}\right) du$ accounts for edge effects ($K_h(v) \approx 1$ if v is far from the border of S).

Remark: $\int_S \hat{\rho}(u) du = n(\mathbf{x})$ in agreement with $\int_S \rho(u) du = \mathbb{E}(n(\mathbf{x}))$.

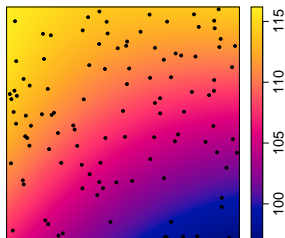
The choice of h is crucial: adaptive choices are available.

- if that makes sense, we can assume a parametric form for $\rho(u)$, i.e. $\rho(u) = \rho_\theta(u)$, and estimate the parameter θ by specific methods.

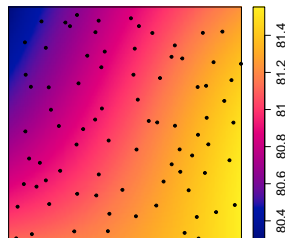
Ex: $\rho_\theta(u) = \exp(\theta_1 z_1(u) + \theta_2 z_2(u))$ where z_1, z_2 are auxiliary variables.

First order intensity ρ : estimation

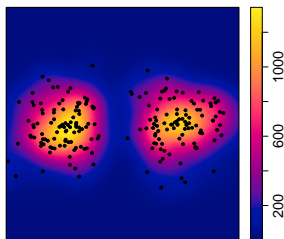
Examples of non-parametric estimation of the intensity



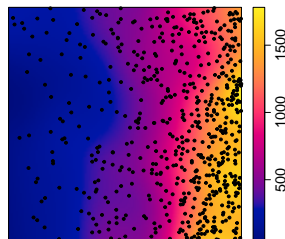
Homogeneous ($\rho = 100$)



Homogeneous ($\rho = 81$)



Inhomogeneous



Inhomogeneous

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

Second order moments : the second order intensity

Aim: analyse the **interaction** between the points (\approx spatial covariance)

Second order moments : the second order intensity

Aim: analyse the **interaction** between the points (\approx spatial covariance)

The **second order intensity** $\rho^{(2)}(u, v)$ satisfies

$$\forall A, B \subseteq S, \quad \mathbb{E} \left(\sum_{\substack{\neq \\ u, v \in \mathbf{X}}} \mathbf{1}_{u \in A, v \in B} \right) = \int_A \int_B \rho^{(2)}(u, v) du dv.$$

Intuitively, $\rho^{(2)}(u, v) \approx P(\mathbf{X} \text{ has a point at } u \text{ and a point at } v)$.

Second order moments : the second order intensity

Aim: analyse the **interaction** between the points (\approx spatial covariance)

The **second order intensity** $\rho^{(2)}(u, v)$ satisfies

$$\forall A, B \subseteq S, \quad \mathbb{E} \left(\sum_{\substack{\neq \\ u, v \in X}} \mathbf{1}_{u \in A, v \in B} \right) = \int_A \int_B \rho^{(2)}(u, v) du dv.$$

Intuitively, $\rho^{(2)}(u, v) \approx P(\mathbf{X} \text{ has a point at } u \text{ and a point at } v)$.

Interpretation:

- If the points are located independently of each other, then

$$\rho^{(2)}(u, v) = \rho(u)\rho(v)$$

- If there is a positive dependence (attraction), then

$$\rho^{(2)}(u, v) > \rho(u)\rho(v)$$

- If there is a negative dependence (inhibition), then

$$\rho^{(2)}(u, v) < \rho(u)\rho(v)$$

Second order moments: the pcf

In spatial statistics, we rather use the **pair correlation function** (pcf) g

$$g(u, v) = \frac{\rho^{(2)}(u, v)}{\rho(u)\rho(v)}.$$

- If $g(u, v) = 1$, there is no interaction between u and v ,
- If $g(u, v) > 1$, there is attraction,
- If $g(u, v) < 1$, there is inhibition.

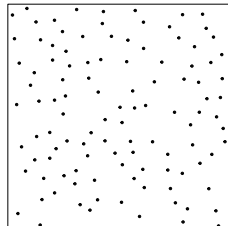
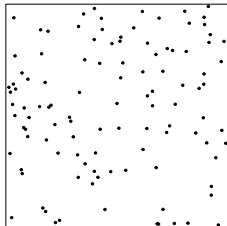
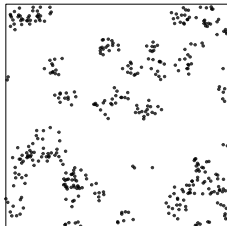
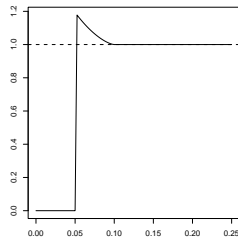
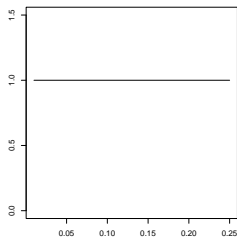
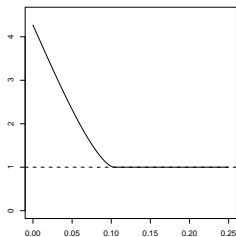
In most applications, it is assumed that $g(u, v)$ only depends on $\|u - v\|$. We then simply define for $r > 0$

$$g(r) = \frac{\rho^{(2)}(u, v)}{\rho(u)\rho(v)} \quad \text{for } \|u - v\| = r.$$

This is the case for a stationary and isotropic point process.

Second order moments: the pcf

Examples of theoretical pcf g , with one realisation for each.



Second order moments: Ripley's K function

Assume that \mathbf{X} is stationary and isotropic, with intensity ρ .

The **Ripley's K function** is defined for any $r > 0$ by

$\rho K(r)$ = expected number of neighbours within distance r of a point $u \in \mathbf{X}$.

Second order moments: Ripley's K function

Assume that \mathbf{X} is stationary and isotropic, with intensity ρ .

The **Ripley's K function** is defined for any $r > 0$ by

$\rho K(r)$ = expected number of neighbours within distance r of a point $u \in \mathbf{X}$.

Mathematically, denoting $B(u, R)$ the ball centred at u with radius R ,

$$\begin{aligned} K(r) &= \frac{1}{\rho} \mathbb{E} \left(n(\mathbf{X} \cap B(u, r)) - 1 \mid u \in \mathbf{X} \right), \\ &= \frac{1}{\rho} \mathbb{E} \left(n(\mathbf{X} \cap B(0, r)) - 1 \mid 0 \in \mathbf{X} \right) \quad \text{by stationarity of } \mathbf{X}. \end{aligned}$$

Second order moments: Ripley's K function

Assume that \mathbf{X} is stationary and isotropic, with intensity ρ .

The **Ripley's K function** is defined for any $r > 0$ by

$\rho K(r)$ = expected number of neighbours within distance r of a point $u \in \mathbf{X}$.

Mathematically, denoting $B(u, R)$ the ball centred at u with radius R ,

$$\begin{aligned} K(r) &= \frac{1}{\rho} \mathbb{E} \left(n(\mathbf{X} \cap B(u, r)) - 1 \mid u \in \mathbf{X} \right), \\ &= \frac{1}{\rho} \mathbb{E} \left(n(\mathbf{X} \cap B(0, r)) - 1 \mid 0 \in \mathbf{X} \right) \quad \text{by stationarity of } \mathbf{X}. \end{aligned}$$

We can prove that

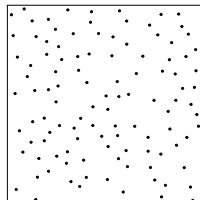
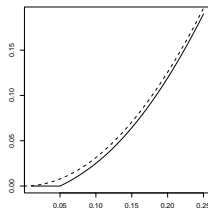
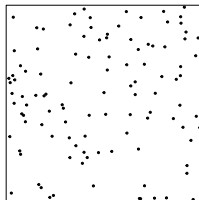
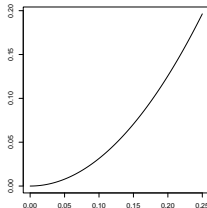
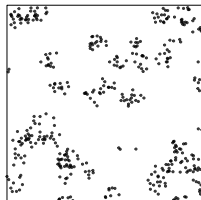
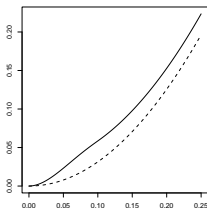
$$K(r) = \int_{B(0,r)} g(\|u\|) du = d\omega_d \int_0^r t^{d-1} g(t) dt,$$

where $\omega_d = |B(0, 1)| = \pi^{d/2} / \Gamma(1 + d/2)$.

Second order moments: Ripley's K function

Remember: $g(r) = 1 \Rightarrow$ no interaction. So:

$$\begin{cases} K(r) = \omega_d r^d \Rightarrow \text{no interaction,} \\ K(r) > \omega_d r^d \Rightarrow \text{attraction,} \\ K(r) < \omega_d r^d \Rightarrow \text{repulsion.} \end{cases}$$



Second order moments: estimation of $K(r)$

Second order moments: estimation of $K(r)$

Since

$\rho K(r)$ = expected number of neighbours within distance r of a point $u \in \mathbf{X}$,

a natural estimator for $K(r)$ based on a realisation \mathbf{x} is

$$\tilde{K}(r) = \frac{1}{\hat{\rho}} \frac{1}{n(\mathbf{x})} \sum_{u \in \mathbf{x}} [n(\mathbf{x} \cap B(u, r)) - 1] = \frac{1}{\hat{\rho}} \frac{1}{n(\mathbf{x})} \sum_{u, v \in \mathbf{x}}^{\neq} \mathbf{1}_{\|u-v\| < r}.$$

Second order moments: estimation of $K(r)$

Since

$\rho K(r)$ = expected number of neighbours within distance r of a point $u \in \mathbf{X}$,

a natural estimator for $K(r)$ based on a realisation \mathbf{x} is

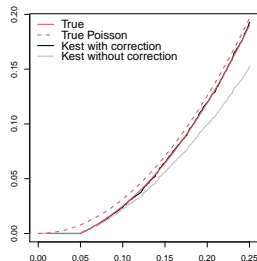
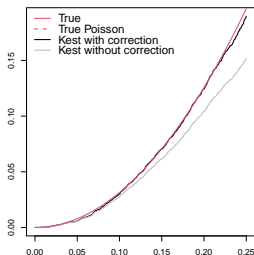
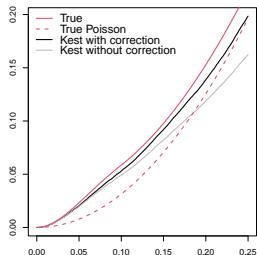
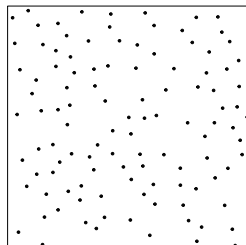
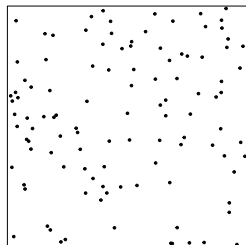
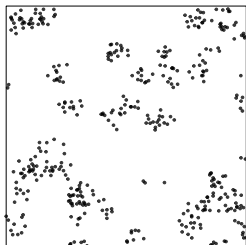
$$\tilde{K}(r) = \frac{1}{\hat{\rho}} \frac{1}{n(\mathbf{x})} \sum_{u \in \mathbf{x}} [n(\mathbf{x} \cap B(u, r)) - 1] = \frac{1}{\hat{\rho}} \frac{1}{n(\mathbf{x})} \sum_{u, v \in \mathbf{x}}^{\neq} \mathbf{1}_{\|u-v\| < r}.$$

But there are edge effects! One solution is to use:

$$\hat{K}(r) = \frac{1}{\hat{\rho}^2} \sum_{u, v \in \mathbf{x}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|},$$

where $S_{u-v} = S + (u - v)$ is the translation of S by $(u - v)$.

Second order moments: estimation of $K(r)$



Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E} \left(\hat{\rho}^2 \hat{K}(r) \right) = \rho^2 K(r)$.

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E} \left(\hat{\rho}^2 \hat{K}(r) \right) = \rho^2 K(r)$.

$$\mathbb{E} \left(\hat{\rho}^2 \hat{K}(r) \right) = \mathbb{E} \left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \right)$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E} \left(\hat{\rho}^2 \hat{K}(r) \right) = \rho^2 K(r)$.

$$\begin{aligned} \mathbb{E} \left(\hat{\rho}^2 \hat{K}(r) \right) &= \mathbb{E} \left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \right) \\ (\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \end{aligned}$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) = \rho^2 K(r)$.

$$\begin{aligned}\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) &= \mathbb{E}\left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|}\right) \\ (\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \\ &= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{v \in S} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} g(\|u - v\|) du dv\end{aligned}$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) = \rho^2 K(r)$.

$$\begin{aligned}\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) &= \mathbb{E}\left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|}\right) \\(\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \\&= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{v \in S} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} g(\|u - v\|) du dv \\(w = u - v) \quad &= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{u-w \in S} \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) du dw\end{aligned}$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) = \rho^2 K(r)$.

$$\begin{aligned}\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) &= \mathbb{E}\left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|}\right) \\(\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \\&= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{v \in S} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} g(\|u - v\|) du dv \\(w = u - v) \quad &= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{u-w \in S} \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) du dw \\&= \rho^2 \int \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) \left(\int \mathbf{1}_{u \in S \cap S_w} du \right) dw\end{aligned}$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) = \rho^2 K(r)$.

$$\begin{aligned}\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) &= \mathbb{E}\left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|}\right) \\(\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \\&= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{v \in S} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} g(\|u - v\|) du dv \\(w = u - v) \quad &= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{u-w \in S} \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) du dw \\&= \rho^2 \int \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) \left(\int \mathbf{1}_{u \in S \cap S_w} du\right) dw \\&= \rho^2 \int \mathbf{1}_{\|w\| < r} g(\|w\|) \frac{|S \cap S_w|}{|S \cap S_w|} dw\end{aligned}$$

Second order moments: estimation of $K(r)$

Why is it a good correction? Let us prove that $\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) = \rho^2 K(r)$.

$$\begin{aligned}\mathbb{E}(\hat{\rho}^2 \hat{K}(r)) &= \mathbb{E}\left(\sum_{u,v \in \mathbf{X}}^{\neq} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|}\right) \\(\text{by def of } \rho^{(2)}) \quad &= \int_S \int_S \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} \rho^{(2)}(u, v) du dv \\&= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{v \in S} \frac{\mathbf{1}_{\|u-v\| < r}}{|S \cap S_{u-v}|} g(\|u - v\|) du dv \\(w = u - v) \quad &= \rho^2 \int \int \mathbf{1}_{u \in S} \mathbf{1}_{u-w \in S} \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) du dw \\&= \rho^2 \int \frac{\mathbf{1}_{\|w\| < r}}{|S \cap S_w|} g(\|w\|) \left(\int \mathbf{1}_{u \in S \cap S_w} du\right) dw \\&= \rho^2 \int \mathbf{1}_{\|w\| < r} g(\|w\|) \frac{|S \cap S_w|}{|S \cap S_w|} dw \\&= \rho^2 K(r).\end{aligned}$$

Second order moments: estimation of $g(r)$

Second order moments: estimation of $g(r)$

Assume again that \mathbf{X} is stationary and isotropic, with intensity ρ .

Remember that $g(r) = \rho^{(2)}(u, v) / \rho^2$ where $r = \|u - v\|$.

For a kernel k (e.g. a Gaussian density) and a bandwidth $h > 0$,

$$\hat{g}(r) = \frac{1}{\hat{\rho}^2} \frac{1}{d\omega_d r^{d-1}} \sum_{u,v \in \mathbf{x}}^{\neq} \frac{1}{h^d} k\left(\frac{\|v - u\| - r}{h}\right) \frac{1}{|S \cap S_{u-v}|}$$

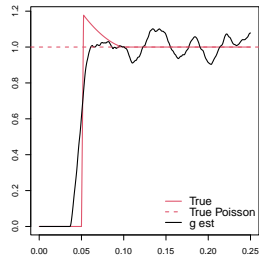
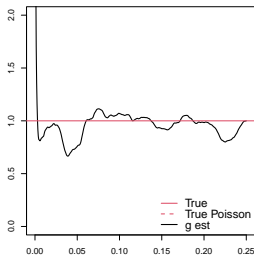
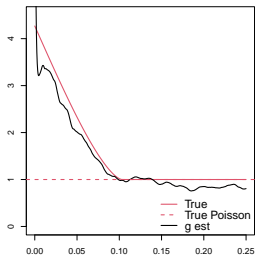
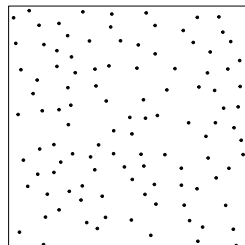
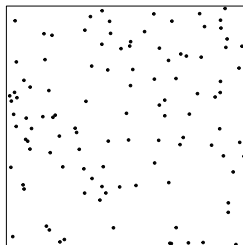
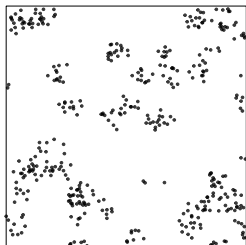
Red term: due to polar coordinates ($\omega_d = |B(0, 1)|$)

Blue term: same edge correction as before

The sum: we are “counting” the number of pairs (u, v) at distance $\approx r$.

As for all non-parametric method, the choice of h is crucial.

Second order moments: estimation of $g(r)$



1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

Assume again that \mathbf{X} is stationary and isotropic, with intensity ρ .

- The L -function: $L(r) = (K(r)/\omega_d)^{1/d}$, with $\omega_d = |B(0, 1)|$.
- The empty space function

$$F(r) = \mathbb{P}(n(\mathbf{X} \cap B(0, r)) > 0).$$

- The nearest neighbour distribution function

$$G(r) = \mathbb{P}(n(\mathbf{X} \cap B(0, r)) > 1 \mid 0 \in \mathbf{X}).$$

- The J -function: $J(r) = (1 - G(r))/(1 - F(r))$.

Interpretation:

No interaction : $L_0(r) = r$, $F_0(r) = G_0(r) = 1 - e^{-\rho\omega_d r^d}$ and $J_0(r) = 1$.

Attraction if $L(r) > r$, $F(r) < F_0(r)$, $G(r) > G_0(r)$, $J(r) < 1$.

Inhibition if $L(r) < r$, $F(r) > F_0(r)$, $G(r) < G_0(r)$, $J(r) > 1$.

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

For each of the considered models, the main questions are:

- What are its main characteristics: attraction? inhibition?
- Do we know its moments (e.g. ρ and g)?
- How can we simulate a realisation?
- How can we fit this model to a point pattern?

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- Short summary

Homogeneous Poisson point process

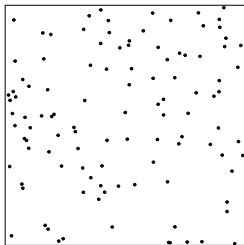
A homogeneous PPP \mathbf{X} with intensity ρ on S satisfies:

- $n(\mathbf{X}) \sim \mathcal{P}(\rho|S|)$, i.e.

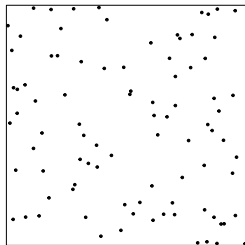
$$P(n(\mathbf{X}) = k) = e^{-\rho|S|} \frac{(\rho|S|)^k}{k!}, \quad \forall k = 0, 1, 2, \dots,$$

- Given $n(\mathbf{X}) = n$, $\{x_1, \dots, x_n\}$ are **independent** and uniform in S .

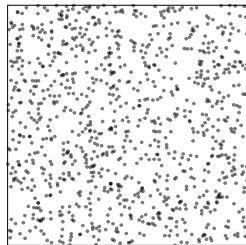
The simulation is straightforward.



$\rho = 100$



$\rho = 100$

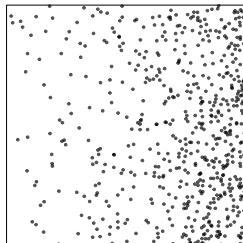
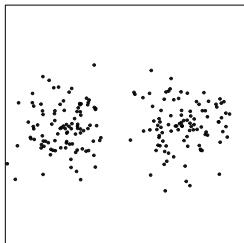


$\rho = 1000$

Inhomogeneous Poisson point process

An inhomogeneous PPP with intensity function $\rho(u)$ on S satisfies:

- $n(\mathbf{X}) \sim \mathcal{P}(\int_S \rho(u) du)$,
- Given $n(\mathbf{X}) = n$, $\{x_1, \dots, x_n\}$ are **independently** distributed in S according to the density $u \mapsto \rho(u) / (\int_S \rho(u) du)$.



Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n)$$

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\begin{aligned}\mathbb{E}(n(\mathbf{X} \cap A)) &= \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \\ &= \sum_{n \geq 0} n \int_A \frac{\rho(u)}{c} du e^{-c} \frac{c^n}{n!} \quad \left(\text{since } x_i | n(\mathbf{X}) \text{ are iid } \sim \frac{\rho(u)}{c} \right)\end{aligned}$$

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\begin{aligned}\mathbb{E}(n(\mathbf{X} \cap A)) &= \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \\ &= \sum_{n \geq 0} n \int_A \frac{\rho(u)}{c} du e^{-c} \frac{c^n}{n!} \quad \left(\text{since } x_i | n(\mathbf{X}) \text{ are iid } \sim \frac{\rho(u)}{c} \right) \\ &= \left(\int_A \rho(u) du \right) \sum_{n \geq 1} e^{-c} \frac{c^{n-1}}{(n-1)!} = \int_A \rho(u) du.\end{aligned}$$

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\begin{aligned}\mathbb{E}(n(\mathbf{X} \cap A)) &= \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \\ &= \sum_{n \geq 0} n \int_A \frac{\rho(u)}{c} du e^{-c} \frac{c^n}{n!} \quad \left(\text{since } x_i | n(\mathbf{X}) \text{ are iid } \sim \frac{\rho(u)}{c} \right) \\ &= \left(\int_A \rho(u) du \right) \sum_{n \geq 1} e^{-c} \frac{c^{n-1}}{(n-1)!} = \int_A \rho(u) du.\end{aligned}$$

- The second order intensity function is $\rho^{(2)}(u, v) = \rho(u)\rho(v)$.

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\begin{aligned}\mathbb{E}(n(\mathbf{X} \cap A)) &= \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \\ &= \sum_{n \geq 0} n \int_A \frac{\rho(u)}{c} du e^{-c} \frac{c^n}{n!} \quad \left(\text{since } x_i | n(\mathbf{X}) \text{ are iid } \sim \frac{\rho(u)}{c} \right) \\ &= \left(\int_A \rho(u) du \right) \sum_{n \geq 1} e^{-c} \frac{c^{n-1}}{(n-1)!} = \int_A \rho(u) du.\end{aligned}$$

- The second order intensity function is $\rho^{(2)}(u, v) = \rho(u)\rho(v)$.
- Hence $g(r) = 1$ for all $r > 0$ and $K(r) = d\omega_d r^d$.

Poisson point process: moments

For a PPP with intensity function $\rho(u)$:

- The intensity function is indeed $\rho(u)$.

We need to prove that for any $A \subseteq S$, $\mathbb{E}(n(\mathbf{X} \cap A)) = \int_A \rho(u) du$.

Denoting $c = \int_S \rho(u) du$ so that $n(\mathbf{X}) \sim \mathcal{P}(c)$,

$$\begin{aligned}\mathbb{E}(n(\mathbf{X} \cap A)) &= \sum_{n \geq 0} \mathbb{E}(n(\mathbf{X} \cap A) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \\ &= \sum_{n \geq 0} n \int_A \frac{\rho(u)}{c} du e^{-c} \frac{c^n}{n!} \quad \left(\text{since } x_i | n(\mathbf{X}) \text{ are iid } \sim \frac{\rho(u)}{c} \right) \\ &= \left(\int_A \rho(u) du \right) \sum_{n \geq 1} e^{-c} \frac{c^{n-1}}{(n-1)!} = \int_A \rho(u) du.\end{aligned}$$

- The second order intensity function is $\rho^{(2)}(u, v) = \rho(u)\rho(v)$.
- Hence $g(r) = 1$ for all $r > 0$ and $K(r) = d\omega_d r^d$.

**PPP is the default model for non-interacting points,
i.e. Complete Spatial Randomness (CSR).**

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) = \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n)$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\begin{aligned} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n) \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) f(u_1, \dots, u_n) \frac{du_1}{|S|} \dots \frac{du_n}{|S|} e^{-|S|} \frac{|S|^n}{n!} \end{aligned}$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\begin{aligned} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n) \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) f(u_1, \dots, u_n) \frac{du_1}{|S|} \dots \frac{du_n}{|S|} e^{-|S|} \frac{|S|^n}{n!} \\ (c = \int_S \rho(u) du) &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) e^{-c} \prod_{i=1}^n \rho(u_i) du_i \frac{1}{n!} \end{aligned}$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\begin{aligned} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n) \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) f(u_1, \dots, u_n) \frac{du_1}{|S|} \dots \frac{du_n}{|S|} e^{-|S|} \frac{|S|^n}{n!} \\ (c = \int_S \rho(u) du) &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) e^{-c} \prod_{i=1}^n \rho(u_i) du_i \frac{1}{n!} \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) \prod_{i=1}^n \frac{\rho(u_i)}{c} du_i \frac{1}{n!} c^n e^{-c} \end{aligned}$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\begin{aligned} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n) \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) f(u_1, \dots, u_n) \frac{du_1}{|S|} \dots \frac{du_n}{|S|} e^{-|S|} \frac{|S|^n}{n!} \\ (c = \int_S \rho(u) du) &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) e^{-c} \prod_{i=1}^n \rho(u_i) du_i \frac{1}{n!} \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) \prod_{i=1}^n \frac{\rho(u_i)}{c} du_i \frac{1}{n!} c^n e^{-c} \\ &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) \end{aligned}$$

Poisson point process: density (or likelihood)

The reference point process \mathbf{X}_0 is the PPP with intensity $\rho = 1$.

A PPP \mathbf{X} with intensity function $\rho(u)$ admits the density wrt \mathbf{X}_0 :

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Proof:

We need to prove that for any test function h , $\mathbb{E}(h(\mathbf{X})) = \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0))$.

$$\begin{aligned} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0)) &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}_0)f(\mathbf{X}_0) | n(\mathbf{X}_0) = n) \mathbb{P}(n(\mathbf{X}_0) = n) \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) f(u_1, \dots, u_n) \frac{du_1}{|S|} \dots \frac{du_n}{|S|} e^{-|S|} \frac{|S|^n}{n!} \\ (c = \int_S \rho(u) du) &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) e^{-c} \prod_{i=1}^n \rho(u_i) du_i \frac{1}{n!} \\ &= \sum_{n \geq 0} \int_{S^n} h(u_1, \dots, u_n) \prod_{i=1}^n \frac{\rho(u_i)}{c} du_i \frac{1}{n!} c^n e^{-c} \\ &= \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X}) | n(\mathbf{X}) = n) \mathbb{P}(n(\mathbf{X}) = n) = \mathbb{E}(h(\mathbf{X})) \end{aligned}$$

Poisson point process: inference

The only parameter to fit is the intensity function $\rho(u)$.

Poisson point process: inference

The only parameter to fit is the intensity function $\rho(u)$.

Given $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$, the log-likelihood is

$$\log f(\mathbf{x}) = |S| - \int_S \rho(u) du + \sum_{i=1}^{n(\mathbf{x})} \log \rho(x_i)$$

Poisson point process: inference

The only parameter to fit is the intensity function $\rho(u)$.

Given $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$, the log-likelihood is

$$\log f(\mathbf{x}) = |S| - \int_S \rho(u) du + \sum_{i=1}^{n(\mathbf{x})} \log \rho(x_i)$$

- For a homogeneous PPP ($\rho(u) = \rho$), $\hat{\rho} = n(\mathbf{x})/|S|$ is the MLE.

Poisson point process: inference

The only parameter to fit is the intensity function $\rho(u)$.

Given $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$, the log-likelihood is

$$\log f(\mathbf{x}) = |S| - \int_S \rho(u) du + \sum_{i=1}^{n(\mathbf{x})} \log \rho(x_i)$$

- For a homogeneous PPP ($\rho(u) = \rho$), $\hat{\rho} = n(\mathbf{x})/|S|$ is the MLE.
- For a parametric inhomogeneous PPP ($\rho(u) = \rho_\theta(u)$),

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{i=1}^{n(\mathbf{x})} \log \rho_\theta(x_i) - \int_S \rho_\theta(u) du.$$

Poisson point process: inference

The only parameter to fit is the intensity function $\rho(u)$.

Given $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$, the log-likelihood is

$$\log f(\mathbf{x}) = |S| - \int_S \rho(u) du + \sum_{i=1}^{n(\mathbf{x})} \log \rho(x_i)$$

- For a homogeneous PPP ($\rho(u) = \rho$), $\hat{\rho} = n(\mathbf{x})/|S|$ is the MLE.
- For a parametric inhomogeneous PPP ($\rho(u) = \rho_\theta(u)$),

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{i=1}^{n(\mathbf{x})} \log \rho_\theta(x_i) - \int_S \rho_\theta(u) du.$$

- For a general inhomogeneous PPP: non-parametric estimator

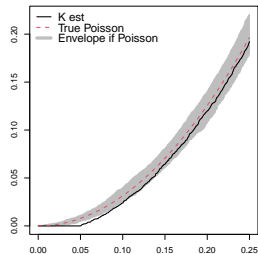
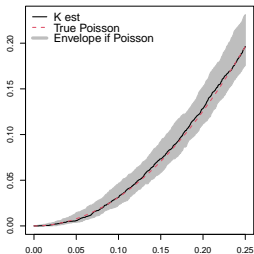
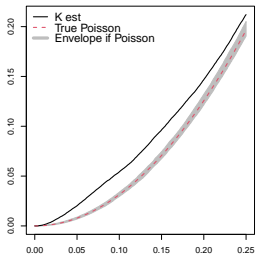
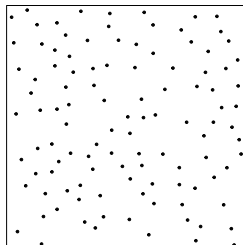
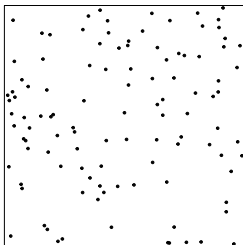
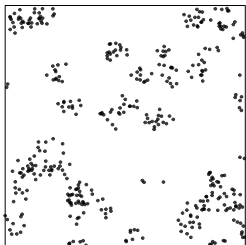
$$\hat{\rho}(u) = \sum_{v \in \mathbf{x}} \frac{1}{h^d} k\left(\frac{v-u}{h}\right) / K_h(v), \quad u \in S.$$

Is a (Poisson) model a good fit to a point pattern?

- 1 Fit the model.
- 2 Simulate many realisations from the fitted model.
- 3 Compute a descriptor for each realisation, for instance $\hat{K}(r)$ or $\hat{g}(r)$.
- 4 Check if the same descriptor for the data is consistent with these simulations.

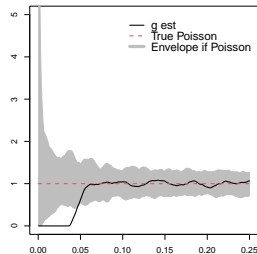
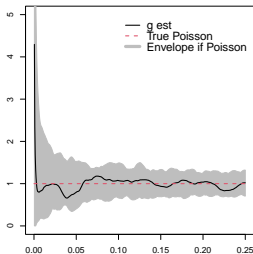
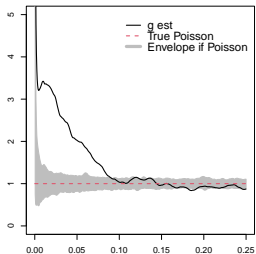
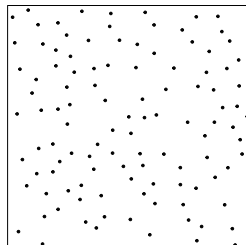
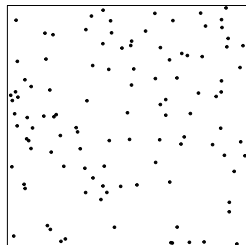
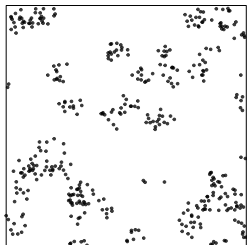
Poisson point process: validation

We fit a homogeneous PPP to these point patterns, and check with K



Poisson point process: validation

idem with g



- 1 Examples of points patterns
- 2 Summary/descriptive statistics
 - First order moment
 - Second order moments
 - Other summary statistics
- 3 Point process models
 - Poisson point process (PPP)
 - Cox point processes
 - Gibbs point processes
 - Determinantal point processes
 - Short summary

Let $\Lambda(u)$, for $u \in S$, be a nonnegative random field on S .

\mathbf{X} is a Cox process if it is a “PPP with random intensity $\Lambda(u)$ ”.

Algorithm of simulation:

- 1 Generate $\Lambda(u)$.
- 2 Given $\Lambda(u) = \lambda(u)$, generate a PPP with intensity $\lambda(u)$.

Cox processes: definition

Let $\Lambda(u)$, for $u \in S$, be a nonnegative random field on S .

\mathbf{X} is a Cox process if it is a “PPP with random intensity $\Lambda(u)$ ”.

Algorithm of simulation:

- 1 Generate $\Lambda(u)$.
- 2 Given $\Lambda(u) = \lambda(u)$, generate a PPP with intensity $\lambda(u)$.

Intuitively, \mathbf{X} will have many points where $\lambda(u)$ takes high values, provided $\lambda(u)$ is smooth enough.

\Rightarrow we expect a clustering behavior

Cox process: example

Example : LGCP (Log Gaussian Cox process)

Let $G(u)$ be a Gaussian random field on S and take

$$\Lambda(u) = \exp(G(u)).$$

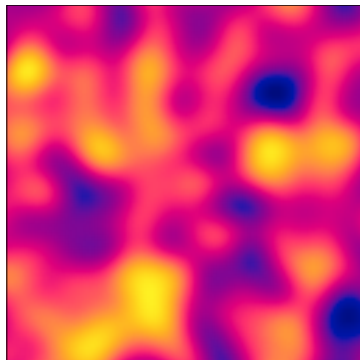
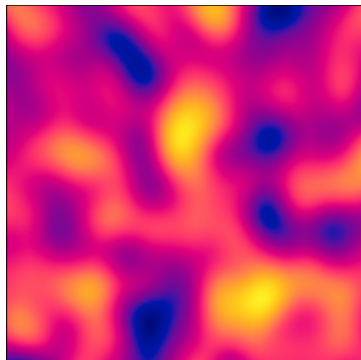
Cox process: example

Example : LGCP (Log Gaussian Cox process)

Let $G(u)$ be a Gaussian random field on S and take

$$\Lambda(u) = \exp(G(u)).$$

- 1 We simulate the Gaussian field $G(u)$



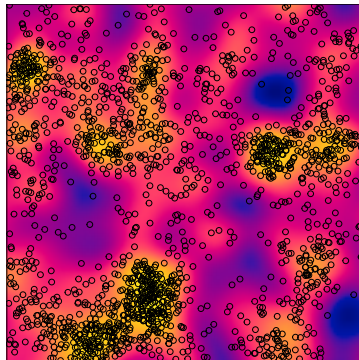
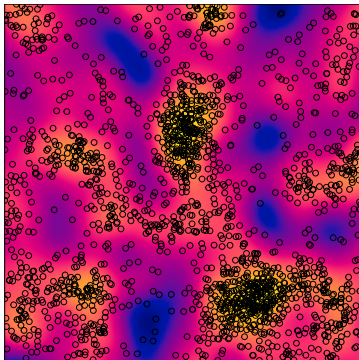
Cox process: example

Example : LGCP (Log Gaussian Cox process)

Let $G(u)$ be a Gaussian random field on S and take

$$\Lambda(u) = \exp(G(u)).$$

- 1 We simulate the Gaussian field $G(u)$
- 2 Given $G(u)$, we generate the PPP with intensity $\exp(G(u))$.



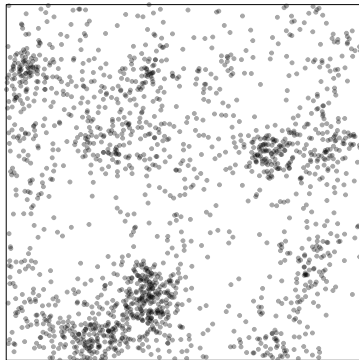
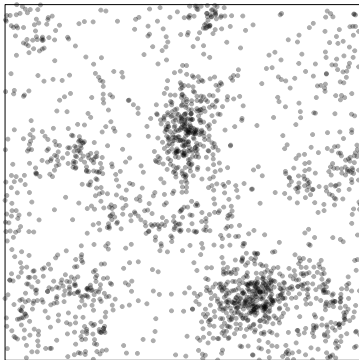
Cox process: example

Example : LGCP (Log Gaussian Cox process)

Let $G(u)$ be a Gaussian random field on S and take

$$\Lambda(u) = \exp(G(u)).$$

- 1 We simulate the Gaussian field $G(u)$
- 2 Given $G(u)$, we generate the PPP with intensity $\exp(G(u))$.



Note the difference with an inhomogeneous PPP:

- for a PPP, the clusters are at the same place for every realisations.
No interaction between the points.
- for a Cox process, the clusters are located randomly.
This creates an **attraction** between the points (see next slide)

Cox process: general properties

- The intensity of a Cox process is $\rho(u) = \mathbb{E}(\Lambda(u))$

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E}(\mathbb{E}(n(\mathbf{X} \cap A) | \Lambda)) = \mathbb{E}\left(\int_A \Lambda(u) du\right) = \int_A \mathbb{E}(\Lambda(u)) du.$$

Cox process: general properties

- The intensity of a Cox process is $\rho(u) = \mathbb{E}(\Lambda(u))$

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E}(\mathbb{E}(n(\mathbf{X} \cap A) | \Lambda)) = \mathbb{E}\left(\int_A \Lambda(u) du\right) = \int_A \mathbb{E}(\Lambda(u)) du.$$

- The second order intensity is $\rho^{(2)}(u, v) = \mathbb{E}(\Lambda(u)\Lambda(v))$.

Cox process: general properties

- The intensity of a Cox process is $\rho(u) = \mathbb{E}(\Lambda(u))$

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E}(\mathbb{E}(n(\mathbf{X} \cap A) | \Lambda)) = \mathbb{E}\left(\int_A \Lambda(u) du\right) = \int_A \mathbb{E}(\Lambda(u)) du.$$

- The second order intensity is $\rho^{(2)}(u, v) = \mathbb{E}(\Lambda(u)\Lambda(v))$.
- So the pcf is

$$g(u, v) = \frac{\mathbb{E}(\Lambda(u)\Lambda(v))}{\mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v))}.$$

If Λ is positively correlated (the typical case), there is attraction:

$$\text{Cov}(\Lambda(u), \Lambda(v)) \geq 0 \Leftrightarrow \mathbb{E}(\Lambda(u)\Lambda(v)) \geq \mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v)) \Leftrightarrow g(u, v) \geq 1.$$

Cox process: general properties

- The intensity of a Cox process is $\rho(u) = \mathbb{E}(\Lambda(u))$

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E}(\mathbb{E}(n(\mathbf{X} \cap A) | \Lambda)) = \mathbb{E}\left(\int_A \Lambda(u) du\right) = \int_A \mathbb{E}(\Lambda(u)) du.$$

- The second order intensity is $\rho^{(2)}(u, v) = \mathbb{E}(\Lambda(u)\Lambda(v))$.
- So the pcf is

$$g(u, v) = \frac{\mathbb{E}(\Lambda(u)\Lambda(v))}{\mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v))}.$$

If Λ is positively correlated (the typical case), there is attraction:

$$\text{Cov}(\Lambda(u), \Lambda(v)) \geq 0 \Leftrightarrow \mathbb{E}(\Lambda(u)\Lambda(v)) \geq \mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v)) \Leftrightarrow g(u, v) \geq 1.$$

- The density wrt to \mathbf{X}_0 (PPP with intensity 1) is

$$f(\mathbf{x}) = \mathbb{E}\left(e^{|S| - \int_S \Lambda(u) du} \prod_{i=1}^{n(\mathbf{x})} \Lambda(x_i)\right), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

Cox process: general properties

- The intensity of a Cox process is $\rho(u) = \mathbb{E}(\Lambda(u))$

$$\mathbb{E}(n(\mathbf{X} \cap A)) = \mathbb{E}(\mathbb{E}(n(\mathbf{X} \cap A) | \Lambda)) = \mathbb{E}\left(\int_A \Lambda(u) du\right) = \int_A \mathbb{E}(\Lambda(u)) du.$$

- The second order intensity is $\rho^{(2)}(u, v) = \mathbb{E}(\Lambda(u)\Lambda(v))$.
- So the pcf is

$$g(u, v) = \frac{\mathbb{E}(\Lambda(u)\Lambda(v))}{\mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v))}.$$

If Λ is positively correlated (the typical case), there is attraction:

$$\text{Cov}(\Lambda(u), \Lambda(v)) \geq 0 \Leftrightarrow \mathbb{E}(\Lambda(u)\Lambda(v)) \geq \mathbb{E}(\Lambda(u))\mathbb{E}(\Lambda(v)) \Leftrightarrow g(u, v) \geq 1.$$

- The density wrt to \mathbf{X}_0 (PPP with intensity 1) is

$$f(\mathbf{x}) = \mathbb{E}\left(e^{|S| - \int_S \Lambda(u) du} \prod_{i=1}^{n(\mathbf{x})} \Lambda(x_i)\right), \quad \forall \mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}.$$

None of ρ , $\rho^{(2)}$, g and f is explicit in general for a Cox process!
But some are known for important cases, see next.

Example: Log Gaussian Cox process (LGCP)

Let $G(u)$ be an isotropic Gaussian random field with

- mean m : $\mathbb{E}(G(u)) = m$,
- covariance function $c(r)$: $\text{Cov}(G(u), G(u+v)) = c(\|v\|)$

The LGCP driven by $G(u)$ is the Cox process driven by $\Lambda(u) = \exp(G(u))$

We know its moments:

$$\begin{aligned}\rho &= \exp(m + c(0)/2), \\ g(r) &= \exp(c(r)).\end{aligned}$$

But the density does not simplify.

Example: Neymann-Scott process

A Neymann-Scott process is a union of clusters:

$$\mathbf{X} = \bigcup_{c \in C} \mathbf{X}_c$$

- C (centers) is a homogeneous PPP with intensity γ
- Given $c \in C$, \mathbf{X}_c (a cluster) is a PPP with intensity $\alpha k(u - c)$

Interpretation of parameters:

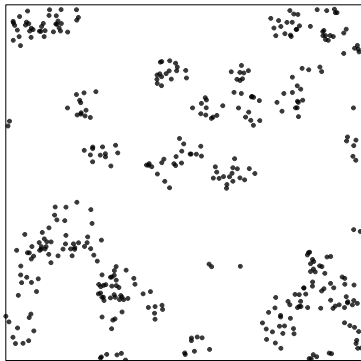
- $\gamma > 0$ is the intensity of clusters,
- $\alpha > 0$ is the mean intensity (or size) of each cluster
- k is a kernel (a density) encoding the spread of each cluster.

This is in fact a Cox point process driven by the random field

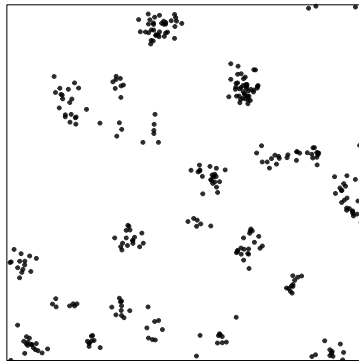
$$\Lambda(u) = \sum_{c \in C} \alpha k(u - c).$$

Example: Neymann-Scott process

Matérn cluster process with radius R : k is the uniform density on $B(0, R)$
Thomas process with variance σ^2 : k is the density of a $\mathcal{N}(0, \sigma^2 I_d)$.



Matérn ($R = 0.05$)



Thomas ($\sigma = 0.02$)

Example: Neymann-Scott process

For a Neymann-Scott process with parameters $\gamma > 0$, $\alpha > 0$ and k .

$$\rho = \gamma\alpha,$$
$$g(r) = 1 + \frac{1}{\gamma} \int k(s)k(s+r)ds.$$

(the integral in g is explicit for a Matérn cluster and a Thomas process)

The density does not simplify.

Cox process: inference

Let a parametric Cox process \mathbf{X} with parameter θ .
For instance $\theta = (\gamma, \alpha, \sigma^2)$ for a Thomas process.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

Cox process: inference

Let a parametric Cox process \mathbf{X} with parameter θ .
For instance $\theta = (\gamma, \alpha, \sigma^2)$ for a Thomas process.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

- Minimum contrast estimation based on g or K , that is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{g}(r)^q - g_{\theta}(r)^q)^2 dr$$

or

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{K}(r)^q - K_{\theta}(r)^q)^2 dr$$

where typically $q = 1/4$, $r_{\min} = 0$ and r_{\max} is to be chosen.

Cox process: inference

Let a parametric Cox process \mathbf{X} with parameter θ .
For instance $\theta = (\gamma, \alpha, \sigma^2)$ for a Thomas process.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

- Minimum contrast estimation based on g or K , that is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{g}(r)^q - g_{\theta}(r)^q)^2 dr$$

or

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{K}(r)^q - K_{\theta}(r)^q)^2 dr$$

where typically $q = 1/4$, $r_{\min} = 0$ and r_{\max} is to be chosen.

- Maximum likelihood estimation

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \mathbb{E} \left(e^{|S| - \int_S \Lambda_{\theta}(u) du} \prod_{i=1}^{n(\mathbf{X})} \Lambda_{\theta}(x_i) \right)$$

but this requires huge Monte-Carlo simulations to approximate $\mathbb{E}(\cdot)$

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

- Poisson point process (PPP)
- Cox point processes
- **Gibbs point processes**
- Determinantal point processes
- Short summary

Gibbs point processes

We speak of a **Gibbs model** when we define \mathbf{X} through its density.

Gibbs point processes

We speak of a **Gibbs model** when we define \mathbf{X} through its density.

Examples :

- A PPP with intensity $\rho(u)$ admits the density (wrt to \mathbf{X}_0):

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i) = c \exp \left(\sum_{u \in \mathbf{x}} \log(\rho(u)) \right)$$

where c does not depend on \mathbf{x} . In the homogeneous case

$$f(\mathbf{x}) = c \exp(\log(\rho) n(\mathbf{x})).$$

We speak of a **Gibbs model** when we define \mathbf{X} through its density.

Examples :

- A PPP with intensity $\rho(u)$ admits the density (wrt to \mathbf{X}_0):

$$f(\mathbf{x}) = e^{|S| - \int_S \rho(u) du} \prod_{i=1}^{n(\mathbf{x})} \rho(x_i) = c \exp \left(\sum_{u \in \mathbf{x}} \log(\rho(u)) \right)$$

where c does not depend on \mathbf{x} . In the homogeneous case

$$f(\mathbf{x}) = c \exp(\log(\rho) n(\mathbf{x})).$$

- We define an homogeneous **pairwise interaction Gibbs model** by

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \sum_{u, v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|) \right),$$

where $\beta > 0$ is related (but not equal) to the log-intensity ρ of \mathbf{x} , and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ is the pairwise interaction function.

Pairwise interaction Gibbs models

Recall : the density wrt to \mathbf{X}_0 (i.e., the PPP with $\rho = 1$) is

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|) \right).$$

Interpretation:

- If $\Phi = 0$, then we're back to a PPP
- A realisation \mathbf{x} will tend to maximise $\sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|)$

Pairwise interaction Gibbs models

Recall : the density wrt to \mathbf{X}_0 (i.e., the PPP with $\rho = 1$) is

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|) \right).$$

Interpretation:

- If $\Phi = 0$, then we're back to a PPP
- A realisation \mathbf{x} will tend to maximise $\sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|)$

Existence: To have $\mathbb{E}(f(\mathbf{X}_0)) = 1$, we must ensure that

$$0 < c^{-1} := \mathbb{E} \left[\exp \left(\beta n(\mathbf{X}_0) + \sum_{u,v \in \mathbf{X}_0}^{\neq} \Phi(\|u - v\|) \right) \right] < \infty.$$

Pairwise interaction Gibbs models

Recall : the density wrt to \mathbf{X}_0 (i.e., the PPP with $\rho = 1$) is

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|) \right).$$

Interpretation:

- If $\Phi = 0$, then we're back to a PPP
- A realisation \mathbf{x} will tend to maximise $\sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|)$

Existence: To have $\mathbb{E}(f(\mathbf{X}_0)) = 1$, we must ensure that

$$0 < c^{-1} := \mathbb{E} \left[\exp \left(\beta n(\mathbf{X}_0) + \sum_{u,v \in \mathbf{X}_0}^{\neq} \Phi(\|u - v\|) \right) \right] < \infty.$$

- $c^{-1} > 0$ is always true since

$$c^{-1} \geq \mathbb{E}[\dots | n(\mathbf{X}_0) = 0] \mathbb{P}(n(\mathbf{X}_0) = 0) = \mathbb{P}(n(\mathbf{X}_0) = 0) > 0.$$

Pairwise interaction Gibbs models

Recall : the density wrt to \mathbf{X}_0 (i.e., the PPP with $\rho = 1$) is

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|) \right).$$

Interpretation:

- If $\Phi = 0$, then we're back to a PPP
- A realisation \mathbf{x} will tend to maximise $\sum_{u,v \in \mathbf{x}}^{\neq} \Phi(\|u - v\|)$

Existence: To have $\mathbb{E}(f(\mathbf{X}_0)) = 1$, we must ensure that

$$0 < c^{-1} := \mathbb{E} \left[\exp \left(\beta n(\mathbf{X}_0) + \sum_{u,v \in \mathbf{X}_0}^{\neq} \Phi(\|u - v\|) \right) \right] < \infty.$$

- $c^{-1} > 0$ is always true since

$$c^{-1} \geq \mathbb{E}[\dots | n(\mathbf{X}_0) = 0] \mathbb{P}(n(\mathbf{X}_0) = 0) = \mathbb{P}(n(\mathbf{X}_0) = 0) > 0.$$

- For $c^{-1} < \infty$, a sufficient assumption is $\Phi \leq 0$, since then

$$c^{-1} \leq \mathbb{E}[\exp(\beta n(\mathbf{X}_0))] < \infty.$$

Example: the Strauss process

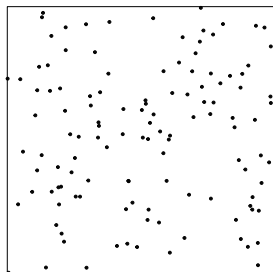
Strauss model with radius $R > 0$ and interaction parameter $\gamma \leq 0$ when

$$\Phi(r) = \gamma \mathbf{1}_{r \leq R}$$

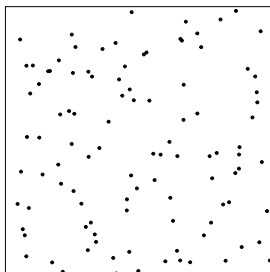
so that

$$f(\mathbf{x}) = c \exp(\beta n(\mathbf{x}) + \gamma s_R(\mathbf{x})),$$

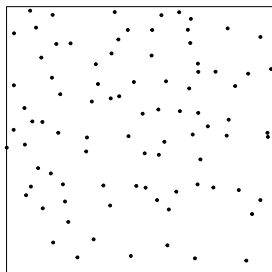
where $s_R(\mathbf{x})$ is the number of R -close pairs of points in \mathbf{x} .



$$\gamma = \log(0.8)$$



$$\gamma = \log(0.5)$$



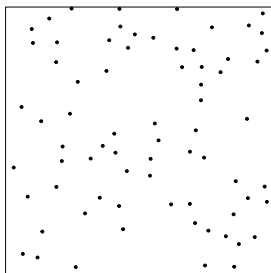
$$\gamma = \log(0.2)$$

Example: the Hardcore point process

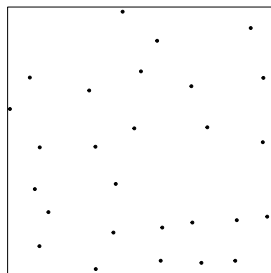
The **Hardcore model** with radius $R > 0$ corresponds to

$$\Phi(r) = \begin{cases} -\infty & \text{if } r \leq R \\ 0 & \text{otherwise} \end{cases}$$

so that no pairs can occur at a distance less than R .



$R = 0.05$



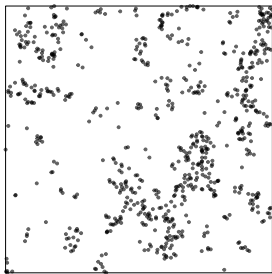
$R = 0.1$

Example: non pairwise interaction

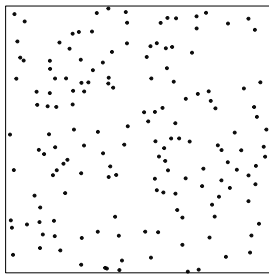
We can consider more complicated density. For instance the **area interaction** process, for $R > 0$ and $\gamma \in \mathbb{R}$,

$$f(\mathbf{x}) = c \exp \left(\beta n(\mathbf{x}) + \gamma \left| \bigcup_{u \in \mathbf{x}} B(u, R) \right| \right),$$

where the interaction depends on the volume of the union of balls.



$\gamma < 0$



$\gamma > 0$

Advantages:

- Flexible models, easy to interpret
- Mainly used to model inhibition between the points
- But some Gibbs models can yield attraction between points

Drawbacks:

- We do not know the moments (neither ρ , nor g , nor K)
- Not easy to simulate (MCMC methods are needed)
- In the density, we do not know explicitly c : this is problematic when it comes to estimate a parameter θ by MLE (since c depends on θ)
- Very hard to deal with in theory

Gibbs point process: Papangelou conditional intensity

The **Papangelou conditional intensity** is defined for any u and \mathbf{x} by

$$\lambda(u, \mathbf{x}) = \frac{f(\mathbf{x} \cup u)}{f(\mathbf{x})}.$$

Intuitively, this is the probability to have a point at u given that the point configuration otherwise is \mathbf{x} .

The benefit of λ over f is that $\lambda(u, \mathbf{x})$ does not depend on c .

Gibbs point process: Papangelou conditional intensity

The **Papangelou conditional intensity** is defined for any u and \mathbf{x} by

$$\lambda(u, \mathbf{x}) = \frac{f(\mathbf{x} \cup u)}{f(\mathbf{x})}.$$

Intuitively, this is the probability to have a point at u given that the point configuration otherwise is \mathbf{x} .

The benefit of λ over f is that $\lambda(u, \mathbf{x})$ does not depend on c .

We can take advantage of that for:

- parametric inference, through the **pseudo-likelihood** estimator:

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{i=1}^{n(\mathbf{x})} \log \lambda_{\theta}(x_i, \mathbf{x} \setminus x_i) - \int_S \lambda_{\theta}(u, \mathbf{x}) du.$$

(note the similarity with the log-likelihood of a PPP)

Gibbs point process: Papangelou conditional intensity

The **Papangelou conditional intensity** is defined for any u and \mathbf{x} by

$$\lambda(u, \mathbf{x}) = \frac{f(\mathbf{x} \cup u)}{f(\mathbf{x})}.$$

Intuitively, this is the probability to have a point at u given that the point configuration otherwise is \mathbf{x} .

The benefit of λ over f is that $\lambda(u, \mathbf{x})$ does not depend on c .

We can take advantage of that for:

- parametric inference, through the **pseudo-likelihood** estimator:

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{i=1}^{n(\mathbf{x})} \log \lambda_{\theta}(x_i, \mathbf{x} \setminus x_i) - \int_S \lambda_{\theta}(u, \mathbf{x}) du.$$

(note the similarity with the log-likelihood of a PPP)

- simulating \mathbf{X} by a birth-death Metropolis-Hastings algorithm, where
 - a birth proposal at u depends (positively) on $\lambda(u, \mathbf{x})$
 - a death proposal of $x_i \in \mathbf{x}$ depends (negatively) on $\lambda(x_i, \mathbf{x} \setminus x_i)$.

- 1 Examples of points patterns
- 2 Summary/descriptive statistics
 - First order moment
 - Second order moments
 - Other summary statistics
- 3 Point process models
 - Poisson point process (PPP)
 - Cox point processes
 - Gibbs point processes
 - **Determinantal point processes**
 - Short summary

Determinantal point processes (DPPs)

The n -th order intensity $\rho^{(n)}$ of a point process satisfies

$$\mathbb{E} \left(\sum_{\substack{\neq \\ u_1, \dots, u_n \in \mathbf{X}}} \mathbf{1}_{u_1 \in A_1, \dots, u_n \in A_n} \right) = \int_{A_1} \cdots \int_{A_n} \rho^{(n)}(u_1, \dots, u_n) du_1 \cdots du_n,$$

$\forall A_1, \dots, A_n \subseteq S$. Intuitively, $\rho^{(n)}(u_1, \dots, u_n) \approx P(\{u_1, \dots, u_n\} \subset \mathbf{X})$.

A DPP is defined through the intensities $\rho^{(n)}$.

Determinantal point processes (DPPs)

The n -th order intensity $\rho^{(n)}$ of a point process satisfies

$$\mathbb{E} \left(\sum_{\substack{\neq \\ u_1, \dots, u_n \in \mathbf{X}}} \mathbf{1}_{u_1 \in A_1, \dots, u_n \in A_n} \right) = \int_{A_1} \cdots \int_{A_n} \rho^{(n)}(u_1, \dots, u_n) du_1 \cdots du_n,$$

$\forall A_1, \dots, A_n \subseteq S$. Intuitively, $\rho^{(n)}(u_1, \dots, u_n) \approx P(\{u_1, \dots, u_n\} \subset \mathbf{X})$.

A DPP is defined through the intensities $\rho^{(n)}$.

Let $C(u, v)$ be a covariance function. \mathbf{X} is a **DPP with kernel** C if

$$\forall n \geq 1, \quad \rho^{(n)}(u_1, \dots, u_n) = \det [C(u_i, u_j)]_{1 \leq i, j \leq n},$$

where $[C(u_i, u_j)]_{1 \leq i, j \leq n}$ denotes the $n \times n$ matrix with entries $C(u_i, u_j)$.

Recall: \mathbf{X} is a DPP with kernel C if

$$\forall n \geq 1, \quad \rho^{(n)}(u_1, \dots, u_n) = \det [C(u_i, u_j)]_{1 \leq i, j \leq n}.$$

If $C(u, v) = C(\|u - v\|)$, then \mathbf{X} is stationary, isotropic and

- its intensity is $\rho^{(1)}(u) = \rho = C(0)$
- its second order intensity $\rho^{(2)}$ is

$$\rho^{(2)}(u, v) = \det \begin{pmatrix} C(0) & C(\|u - v\|) \\ C(\|u - v\|) & C(0) \end{pmatrix} = C(0)^2 - C(\|u - v\|)^2.$$

- Hence its pcf is, for $r = \|u - v\|$, is

$$g(r) = \frac{\rho^{(2)}(u, v)}{\rho(u)\rho(v)} = 1 - \frac{C(r)^2}{C(0)^2}.$$

Since $g \leq 1$, **DPPs are models for inhibitive point processes.**

Stationary DPPs: example

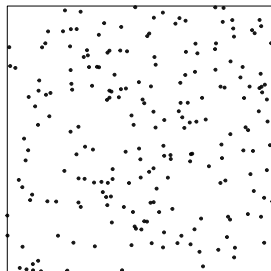
Existence: $C(r)$ must be a covariance function such that $\mathcal{F}(C) \leq 1$.

Stationary DPPs: example

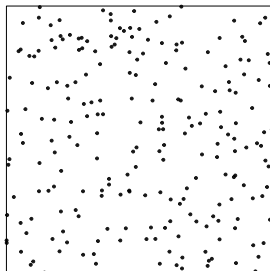
Existence: $C(r)$ must be a covariance function such that $\mathcal{F}(C) \leq 1$.

Example: The Gaussian-type DPP

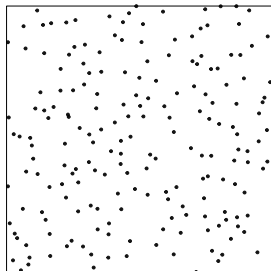
The Gaussian covariance function $C(r) = \rho e^{-r^2/\alpha^2}$ with $\pi\rho\alpha^2 \leq 1$ defines the DPP with intensity ρ and pcf $g(r) = 1 - e^{-2r^2/\alpha^2}$.



small α



medium α



large α

Density of a DPP

In theory, the kernel C admits the eigen-decomposition on S

$$C(u, v) = \sum_{k \geq 1} \lambda_k \Phi_k(u) \Phi_k(v), \quad u, v \in S, \quad (1)$$

where $0 \leq \lambda_k \leq 1$ and $(\Phi_k)_k$ is an orthonormal basis of $L^2(S)$.

Density of a DPP

In theory, the kernel C admits the eigen-decomposition on S

$$C(u, v) = \sum_{k \geq 1} \lambda_k \Phi_k(u) \Phi_k(v), \quad u, v \in S, \quad (1)$$

where $0 \leq \lambda_k \leq 1$ and $(\Phi_k)_k$ is an orthonormal basis of $L^2(S)$.

From this decomposition we can:

- implement a perfect simulation algorithm ;

Density of a DPP

In theory, the kernel C admits the eigen-decomposition on S

$$C(u, v) = \sum_{k \geq 1} \lambda_k \Phi_k(u) \Phi_k(v), \quad u, v \in S, \quad (1)$$

where $0 \leq \lambda_k \leq 1$ and $(\Phi_k)_k$ is an orthonormal basis of $L^2(S)$.

From this decomposition we can:

- implement a perfect simulation algorithm ;
- deduce the density of the DPP when $\lambda_k < 1$, that is

$$f(\mathbf{x}) = c \det [L(x_i, x_j)]_{1 \leq i, j \leq n},$$

where $c = e^{|S|} \prod_{k \geq 1} (1 - \lambda_k)$ and L is another kernel defined by

$$L(u, v) = \sum_{k \geq 1} \frac{\lambda_k}{1 - \lambda_k} \Phi_k(u) \Phi_k(v), \quad u, v \in S.$$

Density of a DPP

In theory, the kernel C admits the eigen-decomposition on S

$$C(u, v) = \sum_{k \geq 1} \lambda_k \Phi_k(u) \Phi_k(v), \quad u, v \in S, \quad (1)$$

where $0 \leq \lambda_k \leq 1$ and $(\Phi_k)_k$ is an orthonormal basis of $L^2(S)$.

From this decomposition we can:

- implement a perfect simulation algorithm ;
- deduce the density of the DPP when $\lambda_k < 1$, that is

$$f(\mathbf{x}) = c \det [L(x_i, x_j)]_{1 \leq i, j \leq n},$$

where $c = e^{|S|} \prod_{k \geq 1} (1 - \lambda_k)$ and L is another kernel defined by

$$L(u, v) = \sum_{k \geq 1} \frac{\lambda_k}{1 - \lambda_k} \Phi_k(u) \Phi_k(v), \quad u, v \in S.$$

But (1) is rarely known, unless C is defined through it.

Approximations exist in the stationary case $C(u, v) = C(\|u - v\|)$.

Let a parametric DPP \mathbf{X} with parameter θ .
For instance $\theta = (\rho, \alpha)$ for a Gaussian-type DPP.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

Let a parametric DPP \mathbf{X} with parameter θ .
For instance $\theta = (\rho, \alpha)$ for a Gaussian-type DPP.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

- Minimum contrast estimation based on g or K , that is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{g}(r)^q - g_{\theta}(r)^q)^2 dr$$

or

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{K}(r)^q - K_{\theta}(r)^q)^2 dr$$

where typically $q = 1/2$, $r_{\min} = 0$ and r_{\max} is to be chosen.

Let a parametric DPP \mathbf{X} with parameter θ .
For instance $\theta = (\rho, \alpha)$ for a Gaussian-type DPP.

We may estimate θ from a realisation \mathbf{x} of \mathbf{X} by:

- Minimum contrast estimation based on g or K , that is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{g}(r)^q - g_{\theta}(r)^q)^2 dr$$

or

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{r_{\min}}^{r_{\max}} (\hat{K}(r)^q - K_{\theta}(r)^q)^2 dr$$

where typically $q = 1/2$, $r_{\min} = 0$ and r_{\max} is to be chosen.

- MLE, if we know the eigen-decomposition of C :

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} f_{\theta}(\mathbf{x}).$$

1 Examples of points patterns

2 Summary/descriptive statistics

- First order moment
- Second order moments
- Other summary statistics

3 Point process models

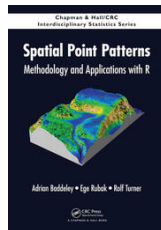
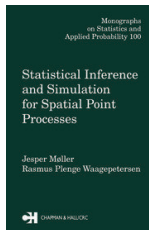
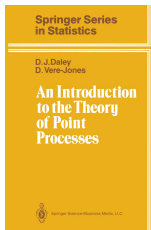
- Poisson point process (PPP)
- Cox point processes
- Gibbs point processes
- Determinantal point processes
- **Short summary**

Summary

Models	Main features	ρ, g	density
Poisson	Only for CSR.	✓	✓
Cox (LGCP, Matérn, Thomas)	Attraction, clusters.	✓	✗
Gibbs	Very flexible. Mainly inhibition but attraction possible.	✗	✓ ¹
DPP	Inhibition. Less flexible than Gibbs.	✓	✓ ²

¹up to the constant; Papangelou conditional density is rather used for inference.

²if we know the eigen-decomposition of the kernel.



- D.J Daley and D. Vere-Jones. *An introduction to the Theory of Point Processes*. Springer Series in Statistics, 1998.
- J. Møller and R. P. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall/CRC, Boca Raton, 2004.
- A. Baddeley, E. Rubak, and R. Turner. *Spatial Point Patterns : Methodology and Applications with R*. CRC Press, 2015.

All illustrations have been implemented with the **spatstat library** in **R**. Almost all presented datasets also come from this library.